Wieler Solenoids from Flat Manifolds
Part 1: The Stable Groupoid

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Outline

- Flat manifolds
- Making Smale spaces from flat manifolds
- The stable groupoid $\mathcal{G}^s(p)$ and its inductive limit structure
- Morita equivalent groupoids
- K-theory of $C^*(\mathcal{G}^s(p))$
- Unstable Putnam homology via groupoid homology of $\mathcal{G}^s(p)$
- Examples
- General results

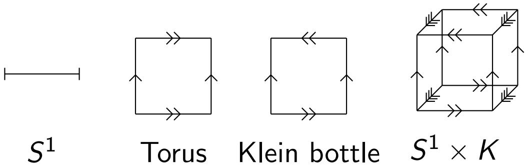
Flat Manifolds

Definition

A flat manifold Y is a closed connected Riemannian d-manifold with zero curvature.

- $Y \cong \frac{\mathbb{R}^d}{\pi_1(Y)}$ and $\pi_1(Y)$ is a torsionfree group of rigid motions of $\mathbb{R}^d$

Examples:



- There exist finitely many flat manifolds in each dimension

Dimension	1	2	3	4	5
Number of flat manifolds	1	2	10	74	1060

Smale spaces from flat manifolds

Theorem (Epstein–Shub)

Let Y be a flat manifold. Then there exists a locally expansive surjection $g : Y \to Y$. Moreover, g is an n-fold cover, $n \geq 2$.

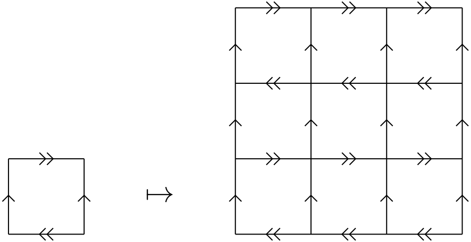
Fix Y, g as above.
Using Williams'/Wieler's construction, we get a Smale space (X, φ):

$$X = \varprojlim(Y, g) = \{(y_n)_{n=0}^{\infty} \mid y_n \in Y \text{ and } g(y_{n+1}) = y_n\}$$
$$\varphi(y_0, y_1, y_2 \ldots) = (g(y_0), g(y_1), g(y_2), \ldots) = (g(y_0), y_0, y_1, \ldots)$$
$$\varphi^{-1}(y_0, y_1, y_2 \ldots) = (y_1, y_2, y_3, \ldots)$$

Examples

(1) The n-solenoid: $Y = S^1 \subseteq \mathbb{C}$ and $g(z) = z^n$.

(2) Let $Y = K$ the Klein bottle and $g : K \to K$ the 9-fold self-cover associated to descending $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ to K.



The stable groupoid

Recall that for a Smale space (X, φ) we define equivalence relations:

(1) stable equivalence: $x \sim_s y$ if $\lim_{n \to \infty} d(\varphi^n(x), \varphi^n(y)) = 0$

(2) unstable equivalence: $x \sim_u y$ if $\lim_{n \to \infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0$

For any $p \in X$ denote $X^u(p) = \{x \in X \mid x \sim_u p\}$.

Theorem (Shub)

$g : Y \to Y$ has a fixed point z.

- So (X, φ) has a fixed point $p = (z, z, z, \dots)$.

Define the stable groupoid for (X, φ) by

$$\mathcal{G}^s(p) = \{(x, y) \in X^u(p) \times X^u(p) \mid x \sim_s y\}.$$

Decomposition of the stable groupoid

Theorem

$x \sim_s y$ if and only if there exists $k \geq 0$ such that $g^k(x_0) = g^k(y_0)$.

Define a subgroupoid of $\mathcal{G}^s(p)$ for each $k \geq 0$

$$\mathcal{G}_k = \left\{ (x, y) \in X^u(p) \times X^u(p) \,\middle|\, g^k(x_0) = g^k(y_0) \right\}.$$

These subgroupoids satisfy:

- $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots$ and the inclusions $i_k : \mathcal{G}_k \hookrightarrow \mathcal{G}_{k+1}$ are open
- Each $\mathcal{G}_k$ is étale
- $\mathcal{G}^s(p) = \displaystyle\bigcup_{k=0}^{\infty} \mathcal{G}_k$
- The inductive limit topology is the usual topology making $\mathcal{G}^s(p)$ étale

Morita equivalent groupoids

$$\mathcal{G}_k = \left\{(x, y) \in X^u(p) \times X^u(p) \,\middle|\, g^k(x_0) = g^k(y_0)\right\}$$

Theorem (Muhley–Renault–Williams)

Let $R \subseteq Z \times Z$ be an equivalence relation on a space Z such that

(i) *the quotient $Z \to Z/R$ is open;*

(ii) *R is closed as a subspace of $Z \times Z$.*

Then $R \sim_{M.e.} Z/R$, where Z/R has the groupoid structure of a space.

Each $\mathcal{G}_k \sim_{M.e.} Y$ since

(i) [DGMW] The projection $\pi_0 : X^u(p) \to Y \cong X^u(p)/\mathcal{G}_0$ defined by $(x, y) \mapsto x_0$ is a cover

(ii) $\mathcal{G}_k$ is clearly closed in $X^u(p) \times X^u(p)$

Theorem (Muhley–Renault–Williams)

Since $\mathcal{G}_k \sim_{M.e.} Y$, we have $C^*(\mathcal{G}_k) \sim_{M.e.} C(Y)$.

K-theory of the stable algebra

We can compute the K-theory of $C^*(\mathcal{G}^s(p))$ as follows.

$$\begin{aligned} K_*(C^*(\mathcal{G}^s(p))) &\cong K_*\left(\varinjlim(C^*(\mathcal{G}_k), i_k)\right) & \mathcal{G}^s(p) = \bigcup_{k\geq 0} \mathcal{G}_k \\ &\cong \varinjlim\left(K_*(C^*(\mathcal{G}_k)), i_{k_*}\right) & \\ &\cong \varinjlim(K^*(Y), g!) & \mathcal{G}^k \sim_{M.e.} Y \end{aligned}$$

where $g! : K^*(Y) \to K^*(Y)$ is the transfer map.

Unstable Putnam homology

We can compute the unstable Putnam homology of (X, φ) as follows.

$$\begin{aligned} H_*^u(X,\varphi) &\cong H_*(\mathcal{G}^s(p)) && \text{[PY]} \\ &\cong \varinjlim (H_*(\mathcal{G}_k), i_{k_*}) && \text{[FKPS]} \\ &\cong \varinjlim (H_*^{\mathcal{G}}(Y), \tilde{g}) && \mathcal{G}_k \sim_{M.e.} Y \\ &\cong \varinjlim (H^*(Y), g!) && \text{[CM]} \end{aligned}$$

where $g! : H^*(Y) \to H^*(Y)$ is the transfer map; in particular,

$$g! \circ g^* = \times n.$$

Example 1: n-solenoid

Let $Y = S^1 \subseteq \mathbb{C}$ and $g : S^1 \to S^1$ be $z \mapsto z^n$.
Recall the transfer map $g! : H^*(S^1) \to H^*(S^1)$ satisfies $g! \circ g^* = \times n$.

$$
\begin{array}{llll}
H^0(S^1) = \mathbb{Z} & g^* = \mathrm{id} & \Rightarrow & g! = \times n \\
H^1(S^1) = \mathbb{Z} & g^* = \times n & \Rightarrow & g! = \mathrm{id}
\end{array}
$$

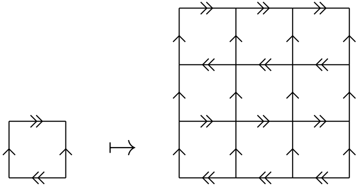
$$H_0^u(X,\varphi) \cong H_0(\mathcal{G}^s(p)) \cong \varinjlim(H^0(S^1), g!) \cong \varinjlim(\mathbb{Z}, \times n) \cong \mathbb{Z}\left[\frac{1}{n}\right]$$

$$H_1^u(X,\varphi) \cong H_1(\mathcal{G}^s(p)) \cong \varinjlim(H^1(S^1), g!) \cong \varinjlim(\mathbb{Z}, \mathrm{id}) \cong \mathbb{Z}$$

Example 2: 9-fold cover of the Klein bottle

$g! \circ g^* = \times 9$

$$\begin{array}{llll} H^0(K) = \mathbb{Z} & g^* = \mathrm{id} & \Rightarrow & g! = \times 9 \\ H^1(K) = \mathbb{Z} & g^* = \times 3 & \Rightarrow & g! = \times 3 \\ H^2(K) = \dfrac{\mathbb{Z}}{2\mathbb{Z}} & g^* = \times 3 \equiv \mathrm{id} & \Rightarrow & g! = \times 3 \equiv \mathrm{id} \end{array}$$



$$H_0^u(X, \varphi) \cong \varinjlim(\mathbb{Z}, \times 9) \cong \mathbb{Z}\left[\frac{1}{9}\right]$$

$$H_1^u(X, \varphi) \cong \varinjlim(\mathbb{Z}, \times 3) \cong \mathbb{Z}\left[\frac{1}{3}\right]$$

$$H_2^u(X, \varphi) \cong \varinjlim\left(\frac{\mathbb{Z}}{2\mathbb{Z}}, \mathrm{id}\right) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

General properties of $H^u_*(-)$

For Y a flat d-manifold and $g : Y \to Y$ an n-fold cover, the following hold for the associated Wieler solenoid (X, φ):

- $H^u_0(X, \varphi) = \varinjlim(\mathbb{Z}, \times n) = \mathbb{Z}\left[\frac{1}{n}\right]$
- If Y is orientable then $H^u_d(X, \varphi) = \mathbb{Z}$

Theorem (Epstein–Shub)

There is a $g : Y \to Y$ so that $T(H_*(\mathcal{G}^s(p))) = T(H^*(Y))$.

Theorem (Charlap)

For each prime p there exists a flat manifold Y with $H_1(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

Corollary

There exists a Smale space (X, φ) such that $\mathbb{Z}/p\mathbb{Z} \subseteq H^u_2(X, \varphi)$.

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